Stresses and stress intensity factor from COD of Vickers indentation cracks

T. FETT
Forschungszentrum Karlsruhe, Institut für Materialforschung II, Postfach 3640, D-76021 Karlsruhe, Germany
E-mail: theo.fett@imf.fzk.de

A. B. KOUNGA, J. RÖDEL
Darmstadt University of Technology, Institute of Materials Science, Petersenstr. 23, 64287 Darmstadt, Germany

The crack opening displacements of Vickers indentations introduced in brittle materials may be evaluated to determine the total stress field and the actual stress intensity factor. The measurable total displacements, $\delta_{\rm total}$, result from the total stresses $\sigma_{\rm total}$ which, in the most general case, summarize residual stresses in the uncracked body $\sigma_{\rm res}$, contact stresses in the inner contact zone $\sigma_{\rm cont}$, bridging stresses $\sigma_{\rm br}$, and, in the case of materials undergoing phase transformation effects, transformation stresses $\sigma_{\rm trans}$ the following equations hold:

$$\sigma_{\rm total} = \sigma_{\rm res} + \sigma_{\rm cont} + \sigma_{\rm br} + \sigma_{\rm trans} \tag{1}$$

and

$$\delta_{\rm total}(r) = \frac{4}{\pi E'}\int_r^a \left(\int_0^{a'} \frac{r'\sigma_{\rm total}(r')}{\sqrt{a'^2 - r'^2}}{\rm d}r'\right)\frac{{\rm d}a'}{\sqrt{a'^2 - r^2}} \tag{2}$$

with the effective Young's modulus E' standing for

$$E' = \begin{cases} E & \text{for plane stress} \\ E/(1-\nu^2) & \text{for plane strain} \end{cases} \tag{3}$$

(E = Young's modulus, ν = Poisson's ratio). For glass the contributions $\sigma_{\rm br}$ and $\sigma_{\rm trans}$ disappear. The shape of displacements visible from the specimen surface is illustrated in Fig. 1b. The crack is open in the region $d < r < a$. In the inner region, $r \le d$, the two crack surfaces are in contact.

Based on the displacements, $\delta_{\rm meas}$, measured by scanning electron microscopy (SEM) or by use of an atomic force microscope (AFM), the total stresses can be determined. In order to obtain $\sigma_{\rm total}$ (r), the integral equation

$$\frac{4}{\pi E'}\int_r^a \left(\int_0^{a'} \frac{r'\sigma_{\rm total}(r')}{\sqrt{a'^2 - r'^2}}{\rm d}r'\right)\frac{{\rm d}a'}{\sqrt{a'^2 - r^2}} = \begin{cases} 0 & \text{for } r < d \\ \delta_{\rm meas} & \text{for } d \le r \end{cases} \tag{4}$$

representing a mixed boundary value problem has to be solved.

If the total stress is determined, the actual stress intensity factor can be determined by

$$K = \frac{2}{\sqrt{\pi a}}\int_0^a \frac{r\sigma_{\rm total}(r)\,{\rm d}r}{\sqrt{a^2 - r^2}} \tag{5}$$

Independently, this value can also be obtained by fitting the Irwin parabola to the near-tip crack opening displacement according to

$$\delta_{\rm near\ tip} \to \sqrt{\frac{8}{\pi}}\frac{K}{E'}\sqrt{a - r} \tag{6}$$

For the evaluation of Equation 4 let us divide the total stress distribution into N equidistant sections of width a/N with N stress values $\sigma_{\rm i}$ constant in the range of $r_{\rm i} < r < r_{\rm i+1}$, as illustrated in Fig. 2. For the displacements at any location with $r < a$ (also in the contact region) resulting from the stress in strip number i, introduction in (2) yields

$$\delta_{\rm i}(r) = \frac{4}{\pi E'}\int_r^a \left(\int_{r_{\rm i}}^{r_{\rm i+1}} \frac{r'\sigma_{\rm i}(r')}{\sqrt{a'^2 - r'^2}}{\rm d}r'\right)\frac{{\rm d}a'}{\sqrt{a'^2 - r^2}}$$
$$= \frac{4\sigma_{\rm i}}{\pi E'}\int_r^a \left(\frac{\sqrt{a'^2 - r_{\rm i}^2} - \sqrt{a'^2 - r_{\rm i+1}^2}}{\sqrt{a'^2 - r^2}}\right){\rm d}a' \tag{7}$$

with the results of

$$\delta_{\rm i}(r) = \frac{4\sigma_{\rm i}}{\pi E'}[G(r, a, r_{\rm i+1}) - G(r, a, r_{\rm i})], \quad r < r_{\rm i} \tag{8a}$$

$$\delta_{\rm i}(r) = \frac{4\sigma_{\rm i}}{\pi E'}[G(r, a, r_{\rm i+1}) - H(r, a, r_{\rm i})], \quad r_{\rm i} \le r < r_{\rm i+1} \tag{8b}$$

$$\delta_{\rm i}(r) = \frac{4\sigma_{\rm i}}{\pi E'}[H(r, a, r_{\rm i+1}) - H(r, a, r_{\rm i})], \quad r_{\rm i+1} \le r \tag{8c}$$

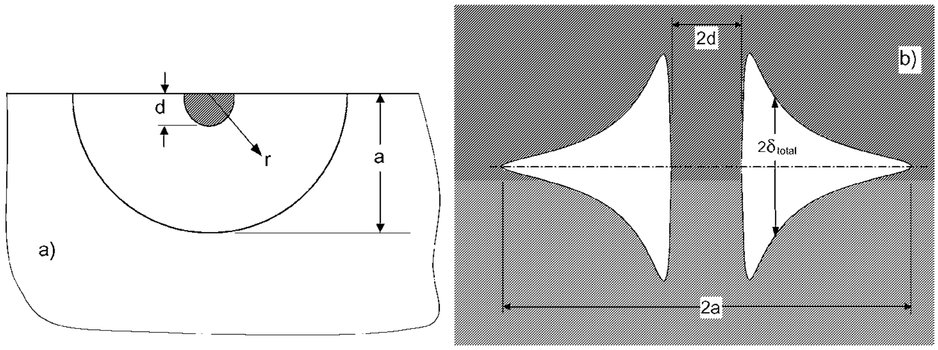

Figure 1 Semi-circular crack introduced by Vickers indentation: (a) view on the cross-section and (b) view on the specimen surface.

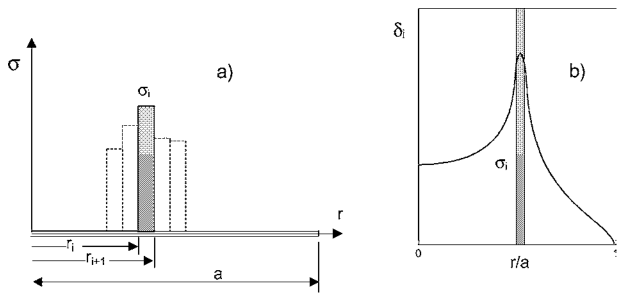

Figure 2 Tesselation of stresses into strips of constant stress: (a) stress and (b) COD caused by stress σ_i.

where the functions G and H are defined as

$$G(r,a,z) = \sqrt{1-\left(\frac{r}{a}\right)^2}\left(1-\sqrt{1-\left(\frac{z}{a}\right)^2}\right) + \frac{z}{a}\left[\mathbf{E}\left(\left(\frac{r}{z}\right)^2\right) - E\left(\arcsin\left(\frac{z}{a},\left(\frac{r}{z}\right)^2\right)\right)\right], \quad r < z \tag{9a}$$

$$\begin{aligned} H(r,a,z) = {} & \sqrt{1-\left(\frac{r}{a}\right)^2}\left(1-\sqrt{1-\left(\frac{z}{a}\right)^2}\right) \\ & + \frac{r}{a}\left[\mathbf{E}\left(\left(\frac{z}{r}\right)^2\right) - E\left(\arcsin\frac{r}{a},\left(\frac{z}{r}\right)^2\right)\right. \\ & -\left(1-\left(\frac{z}{r}\right)^2\right)\left(\mathbf{K}\left(\left(\frac{z}{r}\right)^2\right)\right. \\ & \left.\left. - F\left(\arcsin\frac{r}{a},\left(\frac{z}{r}\right)^2\right)\right)\right], \quad r > z \end{aligned} \tag{9b}$$

Since the complete elliptical integrals **E** and **K** and the incomplete elliptical integrals E and F are available in standard computer programs (e.g., Fortran), numerical treatment of COD does not require any numerical integration. The displacement field for a single σ_i is illustrated in Fig. 2b. The total crack opening displacement at a certain location, resulting from all of the N stress values, is consequently given by superimposing the solutions of Equations 8a–8c.

$$\delta(r) = \sum_{i=1}^{N} \delta_i(r) \tag{10}$$

If a number of M displacements, $\delta_{meas}(r_k)$, has been measured in section $d < r < a$ of the crack width and a number of P disappearing displacements in $r \leq d$ is chosen, a number of $N = M + P$ stress values $\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N$ can be determined in principle by solving the system of linear equations

$$\sum_{j=1}^{M+P} (\delta_{meas}(x_j) - \delta_{comp}(x_j))^2 = 0, \tag{11a}$$

where δ_{comp} are the displacements computed with Equation 10. Having in mind the uncertainties of the individual δ-values, it is recommended to solve an overdetermined system with $M + P > N$ using a least-squares routine

$$\sum_{j=1}^{M+P} (\delta_{meas}(x_j) - \delta_{comp}(x_j))^2 = \min \tag{11b}$$

From the N stress values σ_i known for the whole crack area, the actual stress intensity factor K can be computed simply by

$$K = \frac{2}{\sqrt{\pi a}} \sum_{i=1}^{N} \sigma_i\left(\sqrt{a^2 - r_i^2} - \sqrt{a^2 - r_{i+1}^2}\right) \tag{12}$$

As an example of application, the described procedure may be applied to a (72% SiO_2, 16% CaO, and 12% Na_2O) soda-lime glass ($E = 71$ GPa, $\nu = 0.22$). A Vickers indentation crack was introduced under 50 N load resulting in a crack size of $a = 0.596$ mm. The crack opening displacement (COD) was measured using a SEM. In Fig. 3 COD results are plotted as circles. In order to avoid subcritical crack growth during the measuring time, the specimen was suspended in air for 1 h after the indentation.

In a first step, a smoothing curve was determined from the individual data points. For this purpose, the IMSL "cubic spline data smoother" ICSSCU [1] may be applied. It places a smooth cubic spline along a given set of data points. Treatment by hand using a template also

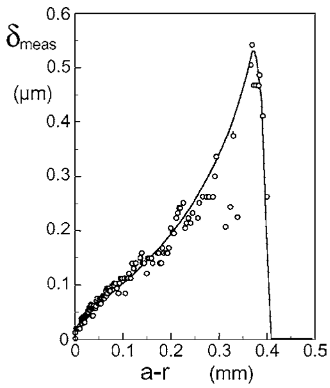

Figure 3 Crack opening displacement measured for a Vickers indentation in soda-lime glass (symbols: measurement, curve: smoothed dependency).

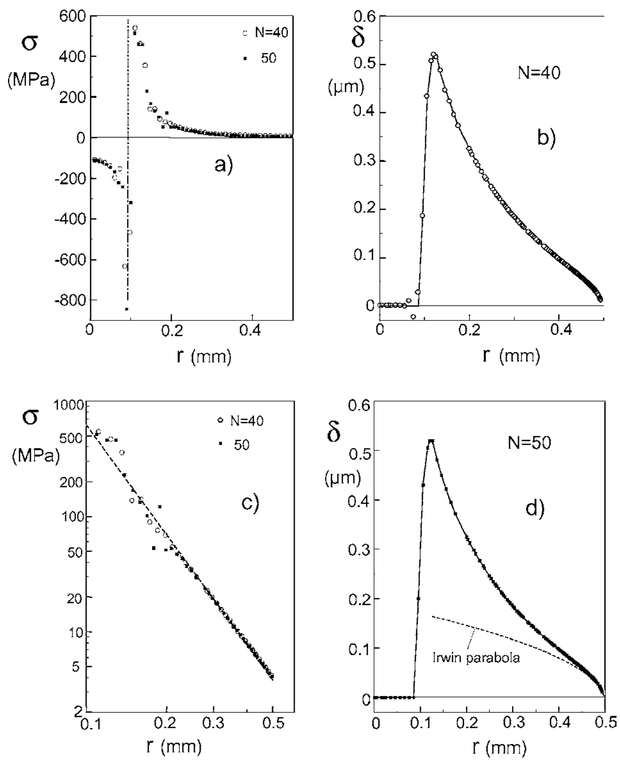

Figure 4 Evaluation of the measurements of Fig. 3: (a) stress distribution obtained for $N = 40$ and 50, (b) comparison of computed displacements with the smoothed CODs for $N = 40$, (c) logarithmic representation of the results in a), computed displacements compared with the smoothed CODs for $N = 50$ (dashed curve: Irwin parabola for $K = 0.38\ \mathrm{MPa}\sqrt{\mathrm{m}}$).

is sufficient. The result is shown by the curve in Fig. 3. At a number of $P = 18$ equidistant locations in the contact area, a disappearing COD was prescribed. The evaluation of Equation 11b was performed with the Harwell subroutine VA02AD [2]. Results reveal a stress representation by a number of $N = 40$ and 50 sections of constant stress, respectively.

In Fig. 4a the resulting stresses are plotted. Circles correspond to $N = 40$, squares to $N = 50$. The dash-dotted line indicates the boundary of the observed contact zone. Fig. 4b and d show the agreement between the COD computed with these stresses and the smoothing curve that was the basis of stress determination. The agreement of the prescribed and the computed crack opening displacements is an indication of the quality of the fit result. For the region of $\delta > 0$, the agreement is excellent in both cases. An increased result is found for $N = 50$ directly at the location where $\delta = 0$ is prescribed for the first time (i.e., at $r \cong 0.09$ mm). Fig. 4c is a logarithmic representation of the results already shown in Fig. 4a for $r \geq 0.1$ mm. The fitting curve entered as the dashed straight line is described by

$$\sigma = 600\,\mathrm{MPa}(r/d)^{-3.16}$$

i.e., very close to the r^{-3} dependency resulting from the cavity model by Hill [3].

From Equation 12, the actual stress intensity factor present during the COD measurements $K = 0.36$ $\mathrm{MPa}\sqrt{\mathrm{m}}$ for $N = 40$ and for $N = 50$ is obtained. By fitting the Irwin parabola according to Equation 6 (see dashed curve in Fig. 4d), $K = 0.38\ \mathrm{MPa}\sqrt{\mathrm{m}}$ is obtained in good agreement with the evaluation via the stresses σ_i. This result is approximately consistent with the subcritical crack growth curve of soda-lime glass as determined by Wiederhorn [4] and Kocer and Collins [5].

In summary, a method is described, which allows to determine the total stress field in the vicinity of a Vickers indentation crack from COD measurements. The numerical procedure is outlined in detail and applied to a crack in soda-lime glass. The stresses in the region of true crack opening are roughly proportional to r^{-3} in accordance with the stress field for a vacancy in an infinite body. The actual stress intensity factor during the COD measurements was found to be $K =$ 0.36–0.38 $\mathrm{MPa}\sqrt{\mathrm{m}}$, depending on the method applied.

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